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ScienceDirect
Journal of Computational and Applied Mathematics 212 (2008) 406–418
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

A collocation method with cubic B-splines for solving the MRLW equation

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Received 17 October 2006; received in revised form 14 December 2006

Abstract

The modified regularized long wave (MRLW) equation is solved numerically by collocation method using cubic B-splines finite element. A linear stability analysis of the scheme is shown to be marginally stable. Three invariants of motion are evaluated to determine the conservation properties of the algorithm, also the numerical scheme leads to accurate and efficient results. Moreover, interaction of two and three solitary waves are studied through computer simulation and the development of the Maxwellian initial condition into solitary waves is also shown.

MSC: 35Q20; 65A30; 65N35

Keywords: MRLW; Collocation method; Solitary waves

1. Introduction

Solitary waves are wave packets or pulses, which propagate in nonlinear dispersive media. Due to dynamical balance between the nonlinear and dispersive effects these waves retain a stable waveform. The regularized long wave (RLW) equation of the form

$$u_t + u_x + \delta u u_x - \mu u_{xxt} = 0, \tag{1}$$

where δ and μ are positive constants, was originally introduced to describe the behavior of the undular bore by Peregrine [10]. This equation is very important in physics media since it describes phenomena with weak nonlinearity and dispersion waves, including nonlinear transverse waves in shallow water, ion-acoustic and magneto hydrodynamic waves in plasma and phonon packets in nonlinear crystals. The solutions of this equation are kinds of solitary waves named solitons whose shapes are not affected by a collision. The RLW equation was solved numerically by various forms of finite element methods [1–6,9,11,13,14] such as Galerkin method, least square method and collocation method with quadratic B-splines, cubic B-splines and recently septic splines. Indeed, the RLW equation is a special case of the

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0377-0427/$ - see front matter © 2007 Published by Elsevier B.V.
doi:10.1016/j.cam.2006.12.029

generalized long wave (GRLW) equation which has the form

$$u_t + u_x + \delta u^p u_x - \mu u_{xxt} = 0, \tag{2}$$

where p is a positive integer. The GRLW equation was studied by few authors, Zhang [15] with finite difference method for a Cauchy problem and Kaya [8] with Adomian decomposition method (ADM). In this paper, we consider another special case of the GRLW called the modified regularized long wave (MRLW) equation. This equation was considered by Gardner et al. [7] using collocation method with quintic B-splines finite element. Here, the collocation method using cubic B-splines is shown to represent accurately the migration of single solitary wave. The interaction of solitary waves and other properties of the MRLW equation are also studied.

2. The governing equation and collocation solution

The MRLW equation as in the form

$$u_t + u_x + 6u^2 u_x - \mu u_{xxt} = 0, \tag{3}$$

where subscripts x and t denote differentiation, and is considered with the boundary conditions $u \to 0$ as $x \to \pm\infty$. In this work, periodic boundary conditions on the region $a \leqslant x \leqslant b$ are assumed in the form

$$\left.\begin{aligned} &u(a,t) = u(b,t) = 0,\\ &u_x(a,t) = u_x(b,t) = 0. \end{aligned}\right\} \tag{4}$$

The exact solution of our equation can be written as [7]

$$u(x,t) = \sqrt{c}\,\sec h(p(x-(c+1)t - x_0)), \tag{5}$$

where $p = \sqrt{\frac{c}{\mu(c+1)}}$, and x_0, c are arbitrary constants.

Also, the Eq. (3) has three invariants as in the form [7]

$$\left.\begin{aligned} &I_1 = \textstyle\int_a^b u\,\mathrm{d}x,\\ &I_2 = \textstyle\int_a^b (u^2 + \mu u_x^2)\,\mathrm{d}x,\\ &I_3 = \textstyle\int_a^b (u^4 - \mu u_x^2)\,\mathrm{d}x. \end{aligned}\right\} \tag{6}$$

Now, partition the interval $[a, b]$ as

$$a = x_0 < x_1 < \cdots < x_N = b, \quad h = x_{j+1} - x_j = \frac{b-a}{N}, \quad j = 0, \ldots, N.$$

Let $\{B_j\}_{j=-1}^{N+1}$ be the cubic B-splines at the knot points x_j; the set of splines form a basis of functions defined over $[a, b]$. A global approximation solution $u^N(x,t)$ is expressed in terms of the cubic B-splines and unknown time dependent parameters as

$$u^N(x,t) = \sum_{j=-1}^{N+1} c_j(t) B_j(x), \tag{7}$$

where c_j are unknown time dependent quantities to be determined from boundary conditions and the collocation conditions. The values of $B_j(x)$ and its derivatives $B'_j(x)$, $B''_j(x)$ at the knots are given in Table 1.

Then, the used nodal values u_j, u'_j and u''_j are given in terms of the parameter c_j by

$$\left.\begin{aligned} &u_j = c_{j-1} + 4c_j + c_{j+1},\\ &u'_j = \frac{3}{h}(c_{j+1} - c_{j-1}),\\ &u''_j = \frac{6}{h^2}(c_{j-1} - 2c_j + c_{j+1}). \end{aligned}\right\} \tag{8}$$

Table 1

x	x_{j-2}	x_{j-1}	x_j	x_{j+1}	x_{j+2}
B_j	0	1	4	1	0
B'_j	0	$\frac{3}{h}$	0	$-\frac{3}{h}$	0
B''_j	0	$\frac{6}{h^2}$	$-\frac{12}{h^2}$	$\frac{6}{h^2}$	0

Now, let us rewrite the Eq. (3) in the form

$$\frac{\partial(u-\mu u_{xx})}{\partial t}+u_x+6u^2u_x=0 \tag{9}$$

and if the time derivative is discretized in a central finite difference fashion $\frac{\partial}{\partial t}u=\frac{u^{n+1}-u^{n-1}}{2k}$, where $k=\Delta t=t_{n+1}-t_n$. Also if we consider $u=u^n$, then Eq. (9) becomes as

$$u^{n+1}-\mu(u_{xx})^{n+1}-u^{n-1}+\mu(u_{xx})^{n-1}+2k((u_x)^n+6(u^2u_x)^n)=0. \tag{10}$$

Introducing Eqs. (8) into Eq. (10) yields

$$a_jc_{j-1}^{n+1}+b_jc_j^{n+1}+a_jc_{j+1}^{n+1}=F_j,\quad j=0,1,\ldots,N. \tag{11}$$

where

$$a_j=1-\frac{6\mu}{h^2},\quad b_j=4+\frac{12\mu}{h^2},\quad F_j=L_3^j-\frac{6\mu}{h^2}L_4^j-2k[6(L_1^j)^2L_2^j+L_2^j],$$

$$L_1^j=c_{j-1}^n+4c_j^n+c_{j+1}^n,\quad L_2^j=\frac{3}{h}(c_{j+1}^n-c_{j-1}^n),$$

$$L_3^j=c_{j-1}^{n-1}+4c_j^{n-1}+c_{j+1}^{n-1},\quad L_4^j=c_{j-1}^{n-1}-2c_j^{n-1}+c_{j+1}^{n-1},\quad j=0,1,\ldots,N.$$

Then this set of equations is a recurrence relationship of element parameters vector $d^n=(c_{-1},c_0,\ldots,c_{N+1})$, so we have $(N+1)$ equations by $(N+3)$ unknown at each level time n, and using the boundary conditions: $u(a,t)=u(b,t)=0$ gives the following relations:

$$\left.\begin{array}{l} c_{-1}=-4c_0-c_1, \\ c_{N+1}=-4c_N-c_{N-1}. \end{array}\right\} \tag{12}$$

Now, by using the Eqs. (12), we can eliminate the variables c_{-1} and c_{N+1}, so the matrix system (11) is reduced to $(N+1)\times(N+1)$ tridiagonal system as follows:

$$\begin{bmatrix} b_0-4a_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ a_1 & b_1 & a_1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_{N-1} & b_{N-1} & a_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & b_N-4a_N \end{bmatrix}\begin{bmatrix} c_0^{n+1} \\ c_1^{n+1} \\ \vdots \\ c_{N-1}^{n+1} \\ c_N^{n+1} \end{bmatrix}=\begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \\ F_N \end{bmatrix}.$$

The above system requires two initial time levels at $t = 0$ and $t = \Delta t = k$, so we use the exact solution (5) to determine the initial conditions d^0 and d^1 as follows:

Firstly, the initial parameter d^0 is determined from the initial and boundary conditions expressed as

$$\left.\begin{aligned} u_x^N(a,0) &= -\frac{3}{h}c_{-1}^0 + \frac{3}{h}c_1^0 = 0,\\ u_x^N(b,0) &= -\frac{3}{h}c_{N-1}^0 + \frac{3}{h}c_{N+1}^0 = 0,\\ u_x^N(x_i,0) &= c_{i-1}^0 + c_i^0 + c_{i+1}^0, \quad i = 0,1,\ldots,N. \end{aligned}\right\} \tag{13}$$

This is a tridiagonal system of equations which is solved with a variant of Thomas algorithms. Also, the parameter d^1 is determined by solving the following system in the same way:

$$\left.\begin{aligned} u_x^N(a,k) &= -\frac{3}{h}c_{-1}^1 + \frac{3}{h}c_1^1 = 0,\\ u_x^N(b,k) &= -\frac{3}{h}c_{N-1}^1 + \frac{3}{h}c_{N+1}^1 = 0,\\ u_x^N(x_i,k) &= c_{i-1}^1 + c_i^1 + c_{i+1}^1, \quad i = 0,1,\ldots,N. \end{aligned}\right\} \tag{14}$$

3. Stability analysis

Like other authors [4,5,7,9,14], our stability analysis will be based on Von Neumann theory in which the growth factor of a typical Fourier mode defined as [12]

$$C_j^n = \hat{\zeta}^n \mathrm{e}^{\mathrm{i}kjh}, \tag{15}$$

where k is a mode number and h is the element size. The nonlinear term u^2u_x cannot be handled by the Fourier mode method, therefore we linearize it. To do this, assume that the quantity u is locally constant. Then substituting Eq. (15) into the system (11) gives

$$\hat{\zeta}^{n+1} = g\hat{\zeta}^n, \tag{16}$$

where g is the growth factor.

So, we have

$$g^2 + 2\mathrm{i}\left(\frac{\gamma\sin\varphi}{2\alpha\cos\varphi + \beta}\right)g - 1 = 0, \tag{17}$$

where $\alpha = (1 - \frac{6\mu}{h^2})$, $\beta = (4 + \frac{12\mu}{h^2})$, $\gamma = 6(\frac{\Delta t}{h} + \frac{6u^2\Delta t}{h})$, and $\varphi = kh$, $i = \sqrt{-1}$.

Lemma. *For h, Δt, φ and u are defined as above*

$$\left|\frac{\gamma\sin\varphi}{2\alpha\cos\varphi + \beta}\right| \leqslant 1, \quad \forall\varphi, \tag{18}$$

where $\alpha = (1 - \frac{6\mu}{h^2})$, $\beta = (4 + \frac{12\mu}{h^2})$, $\gamma = 6(\frac{\Delta t}{h} + \frac{6u^2\Delta t}{h})$.

Proof.

$$\left|\frac{\gamma\sin\varphi}{2\alpha\cos\varphi + \beta}\right| = \left|\frac{3\Delta t(1 + 6u^2)\sin\varphi}{h[2 + \cos\varphi - (6\mu/h^2)(\cos\varphi - 1)]}\right|.$$

Now, suppose $x = 1 - \cos\varphi$, then

$$\sup_{\forall\varphi}\left[\frac{3\Delta t(1+6u^2)\sin\varphi}{h[2+\cos\varphi-(6\mu/h^2)(\cos\varphi-1)]}\right]^2 = \sup_{0\leqslant x\leqslant 2}\frac{9(\Delta t)^2(1+6u^2)^2 x(2-x)}{h^2[3-(1-(6\mu/h^2))x]^2},$$

Suppose that $3-(1-\frac{6\mu}{h^2})x=0$, then $x=\frac{3}{1-6\mu/h^2}$, and so $0\leqslant\frac{3}{1-6\mu/h^2}\leqslant 2$, and we have $1-\frac{6\mu}{h^2}\geqslant\frac{3}{2}\Rightarrow\frac{6\mu}{h^2}\leqslant-\frac{1}{2}$. But μ is a positive constant, so we have a contradiction. Hence $3-(1-\frac{6\mu}{h^2})x\neq 0$. Let $f(x)=\frac{2x-x^2}{(3-rx)^2}$, where $r=1-\frac{6\mu}{h^2}$, then $f(x)$ is continuous on [0, 2]. Now, we differentiate f with respect to x, to find the maximum, then

$$f'(x)=\frac{2(1-x)(3-rx)+2r(2x-x^2)}{(3-rx)^3}=0,$$

so $x=\frac{3}{3-r}$. Now, we have

$$\sup_{0\leqslant x\leqslant 2}\frac{x(2-x)}{[3-rx]^2}=\frac{3/(3-r)(2-3/(3-r))}{[3-r(3/(3-r))]^2}=\frac{3(3-2r)}{(9-6r)^2}=\frac{1}{3(3-2r)}=\frac{h^2}{3(h^2+12\mu)},$$

so, we get

$$\sup_{\forall\varphi}\left[\frac{3\Delta t(1+6u^2)\sin\varphi}{h[2+\cos\varphi-(6\mu/h^2)(\cos\varphi-1)]}\right]^2=\frac{3(\Delta t)^2(1+6u^2)^2}{h^2+12\mu}, \tag{19}$$

The space step length h is usually small, and the time step length Δt is also usually small. Also, u represents single speed and is usually small. So, the following relation will be satisfied for any problem of practical significance:

$$\frac{3(\Delta t)^2(1+6u^2)^2}{h^2+12\mu}\leqslant 1.$$

Hence,

$$\left|\frac{\gamma\sin\varphi}{2\alpha\cos\varphi+\beta}\right|\leqslant 1,\quad\forall\varphi.$$

Next, from the above lemma, we can put

$$\frac{\gamma\sin\varphi}{2\alpha\cos\varphi+\beta}=\sin\theta, \tag{20}$$

for any angle θ.

So, Eq. (17) becomes

$$g^2+2\mathrm{i}g\sin\theta-1=0, \tag{21}$$

Thus, $g_1=\mathrm{e}^{\mathrm{i}\theta}$ and $g_2=-\mathrm{e}^{-\mathrm{i}\theta}$. Therefore, we have

$$|g_1|=1\quad\text{and}\quad|g_2|=1. \tag{22}$$

Showing that our scheme is marginally stable. □

4. Numerical applications

In this section, we present some numerical experiments to find the numerical solutions of single solitary waves, and determining the solution of two and three solitons interaction at different time levels. We also show the development of the Maxwellian initial condition into solitary waves.

Table 2
Invariants and errors for single solitary wave with $c=1$, $h=0.2$, $k=0.025$, $0 \leqslant x \leqslant 100$

Time	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
0	4.44288	3.29982	1.41420	0	0
1	4.44288	3.29985	1.41425	2.39713	1.97253
2	4.44288	3.29984	1.41420	3.77795	2.53315
3	4.44288	3.29983	1.41420	4.61143	2.88226
4	4.44288	3.29983	1.41420	5.29405	3.22690
5	4.44288	3.29983	1.41420	5.94584	3.57898
6	4.44288	3.29983	1.41420	6.59814	3.93587
7	4.44288	3.29983	1.41420	7.25929	4.30829
8	4.44288	3.29983	1.41420	7.93090	4.68346
9	4.44288	3.29983	1.41420	8.61228	5.05992
10	4.44288	3.29983	1.41420	9.30196	5.43718

Table 3
Errors and invariants for single solitary wave with $c=1$, $h=0.2$, $k=0.025$, $0 \leqslant x \leqslant 100$, $t=10$

Schemes	I_1	I_2	I_3	$L_2 \times 10^3$	$L_\infty \times 10^3$
Analytical	4.44288	3.29983	1.41421	0	0
Our scheme	4.44288	3.29983	1.41420	9.30196	5.43718
Cubic B-splines coll-CN [7]	4.442	3.299	1.413	16.39	9.24
Cubic B-splines coll+PA-CN [7]	4.440	3.296	1.411	20.3	11.2

4.1. The motion of single solitary waves

To examine the accuracy and the efficiency of our scheme, we consider two cases in our numerical work. Since L_∞-norm and L_2-norm are used to compare our numerical solutions with the exact solution (5), and the quantities I_1, I_2 and I_3 are evaluated to measure the conservation properties of the collocation scheme, the analytical values of these invariants can be found in [7] as

$$\left.\begin{aligned} I_1 &= \textstyle\int_{-\infty}^{\infty} u(x,0)\,\mathrm{d}x = \frac{\pi\sqrt{c}}{p},\\ I_2 &= \textstyle\int_{-\infty}^{\infty} (u^2(x,0)+\mu u_x^2(x,0))\,\mathrm{d}x = \frac{2c}{p}+\frac{2\mu p c}{3},\\ I_3 &= \textstyle\int_{-\infty}^{\infty} (u^4(x,0)-\mu u_x^2(x,0))\,\mathrm{d}x = \frac{4c^2}{3p}-\frac{2\mu p c}{3}. \end{aligned}\right\} \tag{23}$$

In the first case, the parameters $c=1$, $h=0.2$, $k=0.025$, $\mu=1$ and $x_0=40$ with range [0,100] are chosen to coincide with the cubic schemes [7], then solitary wave has amplitude 1, and the simulations are done up to $t=10$. The analytical values for the invariants are $I_1=4.44288$, $I_2=3.29983$ and $I_3=1.41421$. The invariants I_2 and I_3 change from their initial values by less than 3×10^{-5} and 5×10^{-5} percent, respectively, during the time of running, whereas, the changes of invariant I_1 approach to zero throughout. Thus satisfactory quantities are obtained. Also the error norms L_∞ and L_2 are satisfactorily small at all times, where L_∞-error $=5.43718 \times 10^{-3}$ and L_2-error $=9.30196 \times 10^{-3}$ at $t=10$. All results for the first case are documented in Table 2. Moreover, Table 3 represents the values of the invariants and error norms of the present method at time 10 against the recorded results of Gardner et al. [7]. We find that our scheme provides good results than others. The motion of solitary wave using our collocation scheme is plotted at different time levels in Fig. 1. In case II, we choose $c=0.3$, $h=0.1$, $k=0.01$, $\mu=1$ and $x_0=40$ with range [0,100], then the amplitude is 0.54772. The simulations are done up to $t=20$ and the analytical values of the invariants are $I_1=3.58197$, $I_2=1.34508$ and $I_3=0.153723$. The changes of the invariants I_1, I_2 and I_3 from the initial variants approach to zero throughout and agree with the analytical values for these invariants, which indicated that our scheme is

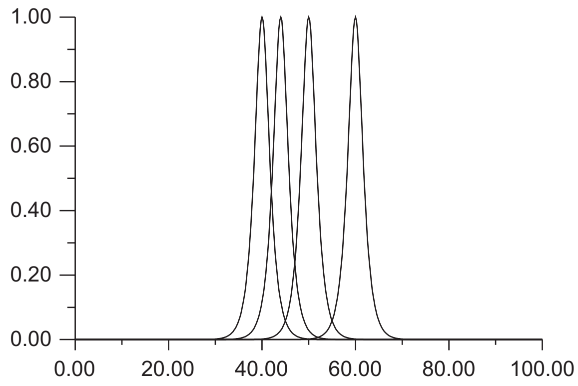

Fig. 1. Single solitary wave with $c = 1$, $x_0 = 40$, $0 \leqslant x \leqslant 100$ at level times: $t = 0, 2, 5$ and 10.

Table 4
Errors and invariants for single solitary wave with $c = 0.3$, $h = 0.1$, $k = 0.01$, $0 \leqslant x \leqslant 100$

Time	I_1	I_2	I_3	$L_2 \times 10^4$	$L_\infty \times 10^4$
0	3.58197	1.34508	0.153723	0	0
2	3.58197	1.34508	0.153723	1.47874	1.03002
4	3.58197	1.34508	0.153723	2.44705	1.46471
6	3.58197	1.34508	0.153723	3.08313	1.69194
8	3.58197	1.34508	0.153723	3.58473	1.88419
10	3.58197	1.34508	0.153723	4.02927	2.06732
12	3.58197	1.34508	0.153723	4.44840	2.24782
14	3.58197	1.34508	0.153723	4.85711	2.42701
16	3.58197	1.34508	0.153723	5.26135	2.60593
18	3.58197	1.34508	0.153723	5.66440	2.78546
20	3.58197	1.34508	0.153723	6.06885	2.96650

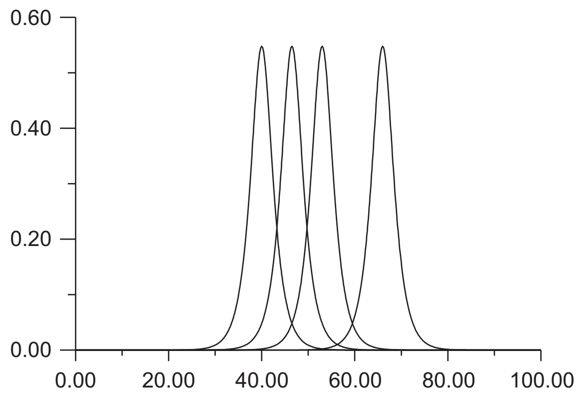

Fig. 2. Single solitary wave with $c = 0.3$, $x_0 = 40$, $0 \leqslant x \leqslant 100$ at level times: $t = 0, 5, 10$ and 20.

satisfactorily conservative. Errors in L_∞-norm and L_2-norm are satisfactorily small, since L_∞-error $= 2.96650 \times 10^{-4}$ and L_2-error $= 6.06885 \times 10^{-4}$at $t = 20$. The results for the second case are shown in Table 4. Fig. 2 illustrates the motion of the solitary wave for this case at different time levels.

Table 5
Invariants for interaction of two solitary waves with $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$, $h = 0.2$, $k = 0.025$, $0 \leqslant x \leqslant 250$

Time	I_1	I_2	I_3
0	11.4677	14.6291	22.8806
2	11.4677	14.6292	22.8807
4	11.4677	14.6292	22.8807
6	11.4677	14.6295	22.8806
8	11.4677	14.6451	22.8454
10	11.4677	14.5963	22.8913
12	11.4677	14.6287	22.8814
14	11.4677	14.6295	22.8807
16	11.4677	14.6294	22.8808
18	11.4677	14.6293	22.8809
20	11.4677	14.6292	22.8809

4.2. Interaction of two solitary waves

In this section, we study the interaction of two MRLW solitary waves having different amplitudes and traveling in the same direction. We consider the MRLW equation with initial conditions given by the linear sum of two well separated solitary waves of various amplitudes

$$u(x,0) = \sum_{i=1}^{2} A_i \sec h(p_i(x - x_i)), \tag{24}$$

where $A_i = \sqrt{c_i}$, $p_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}$, $i = 1, 2$, x_i and c_i are arbitrary constants. From Eqs. (23), the analytical values of the conservation laws of this case can be found as

$$\left.\begin{aligned} I_1 &= \sum_{i=1}^{2} \frac{\pi\sqrt{c_i}}{p_i}, \\ I_2 &= \sum_{i=1}^{2} \left(\frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3}\right), \\ I_3 &= \sum_{i=1}^{2} \left(\frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3}\right). \end{aligned}\right\} \tag{25}$$

In our computational work, we choose $c_1 = 4$, $c_2 = 1$, $x_1 = 25$, $x_2 = 55$, $\mu = 1$, $h = 0.2$, $k = 0.025$ with interval [0,250], then the amplitudes are in ratio 2:1, where $A_1 = 2A_2$. The analytical values of the invariants for this case are $I_1 = 11.467698$, $I_2 = 14.629243$ and $I_3 = 22.880466$, and the changes in I_2 and I_3 are less than 3.32×10^{-2} and 4.59×10^{-2} percent, respectively, whereas, the changes in I_1 approach to zero throughout, as shown in Table 5. Fig. 3 shows the computer plot of the interaction of these solitary waves at different time levels, where the simulation is done up to $t = 20$.

4.3. Interaction of three solitary waves

The interaction of three MRLW solitary waves having different amplitudes and traveling in the same direction is illustrated. We consider the MRLW equation with initial conditions given by the linear sum of three well separated solitary waves of various amplitudes

$$u(x,0) = \sum_{i=1}^{3} A_i \sec h(p_i(x - x_i)), \tag{26}$$

where $A_i = \sqrt{c_i}$, $p_i = \sqrt{\frac{c_i}{\mu(c_i+1)}}$, $i = 1, 2, 3$, x_i and c_i are arbitrary constants.

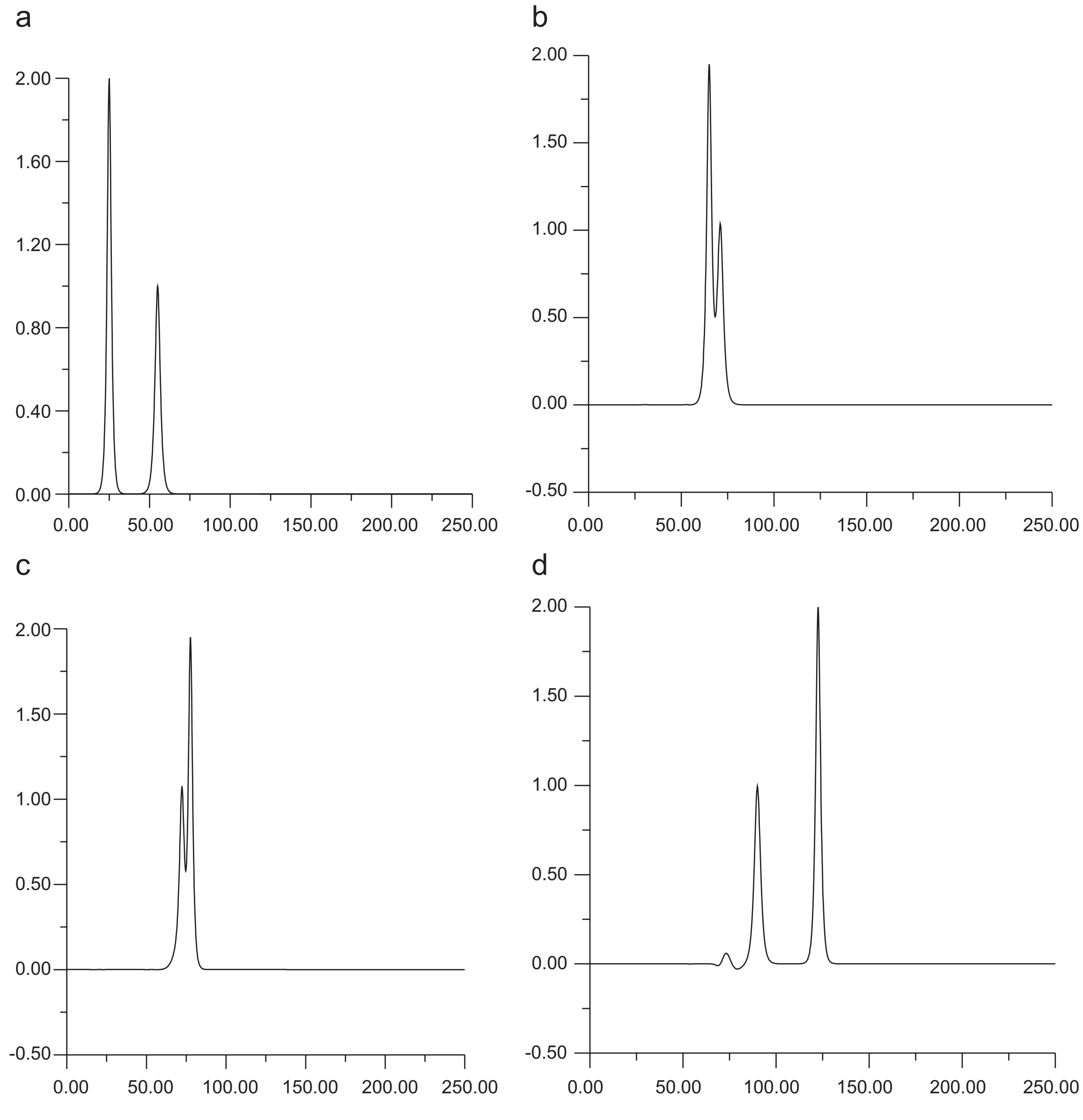

Fig. 3. Interaction of two solitary waves at: (a) $t = 0$, (b) $t = 8$, (c) $t = 10$, (d) $t = 19$.

And from Eqs. (23), the analytical values of the conservation laws of this case can be found as

$$\left.\begin{aligned} I_1 &= \sum_{i=1}^{3} \frac{\pi\sqrt{c_i}}{p_i}, \\ I_2 &= \sum_{i=1}^{3} \left(\frac{2c_i}{p_i} + \frac{2\mu p_i c_i}{3}\right), \\ I_3 &= \sum_{i=1}^{3} \left(\frac{4c_i^2}{3p_i} - \frac{2\mu p_i c_i}{3}\right). \end{aligned}\right\} \tag{27}$$

In our computational work, we choose $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, $\mu = 1$, $h = 0.2$, $k = 0.025$ with interval [0,250], then the amplitudes are in ratio 4:2:1, where $A_1 = 2A_2 = 4A_3$. The analytical values of the invariants

Table 6
Invariants for interaction of three solitary waves with $c_1 = 4$, $c_2 = 1$, $c_3 = 0.25$, $x_1 = 15$, $x_2 = 45$, $x_3 = 60$, $h = 0.2$, $k = 0.025$, $0 \leqslant x \leqslant 250$

Time	I_1	I_2	I_3
0	13.6891	15.4549	22.8816
5	13.6891	15.3109	22.6939
10	13.6891	15.6514	22.8388
15	13.6891	15.6548	22.9347
20	13.6891	15.6557	22.9330
25	13.6892	15.6559	22.9336
30	13.6894	15.6559	22.9348
35	13.6913	15.6564	22.9343
40	13.7015	15.6566	22.9335
45	13.7043	15.6563	22.9303

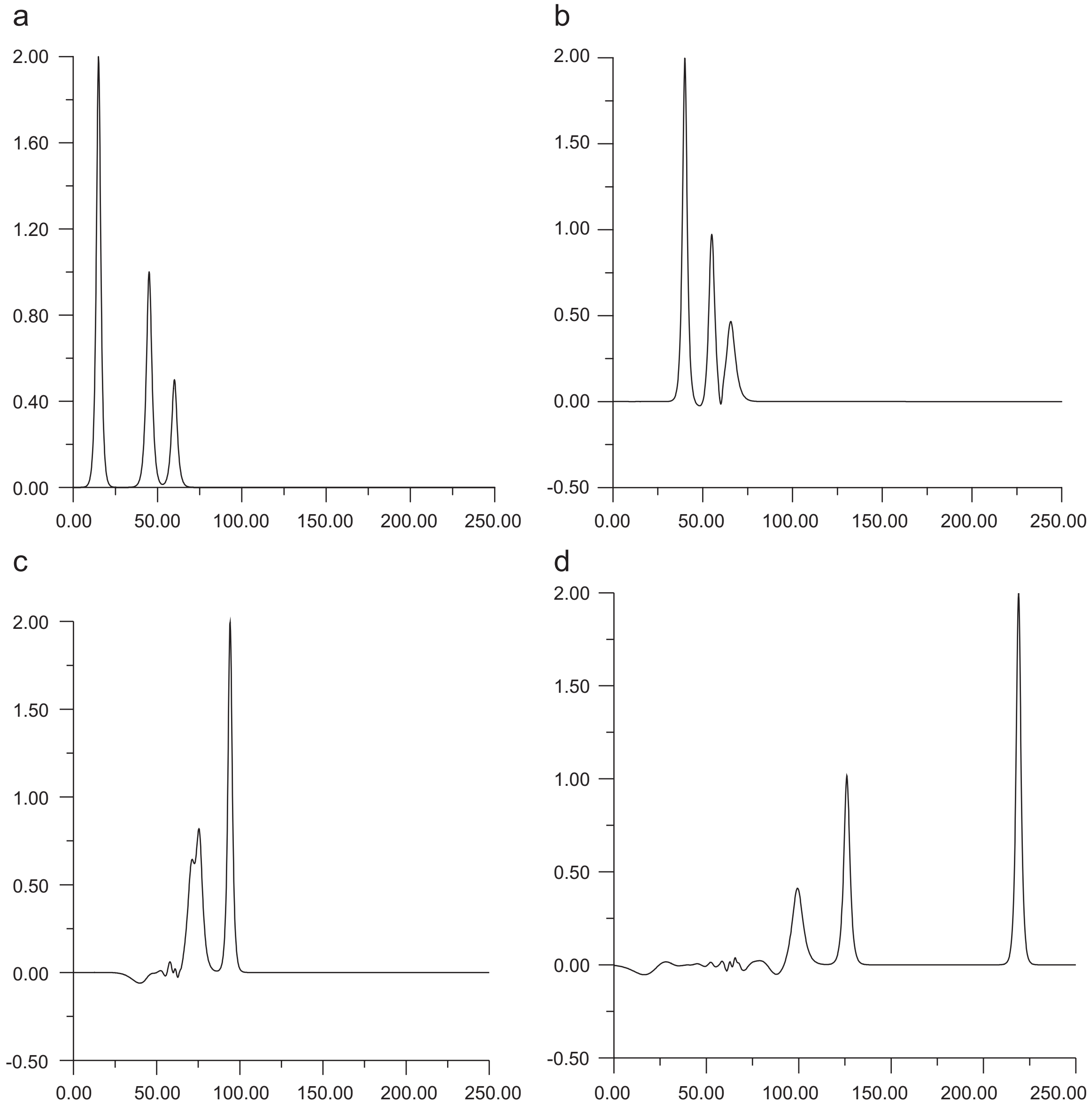

Fig. 4. Interaction of three solitary waves at: (a) $t = 0$, (b) $t = 5$, (c) $t = 15$, (d) $t = 40$.

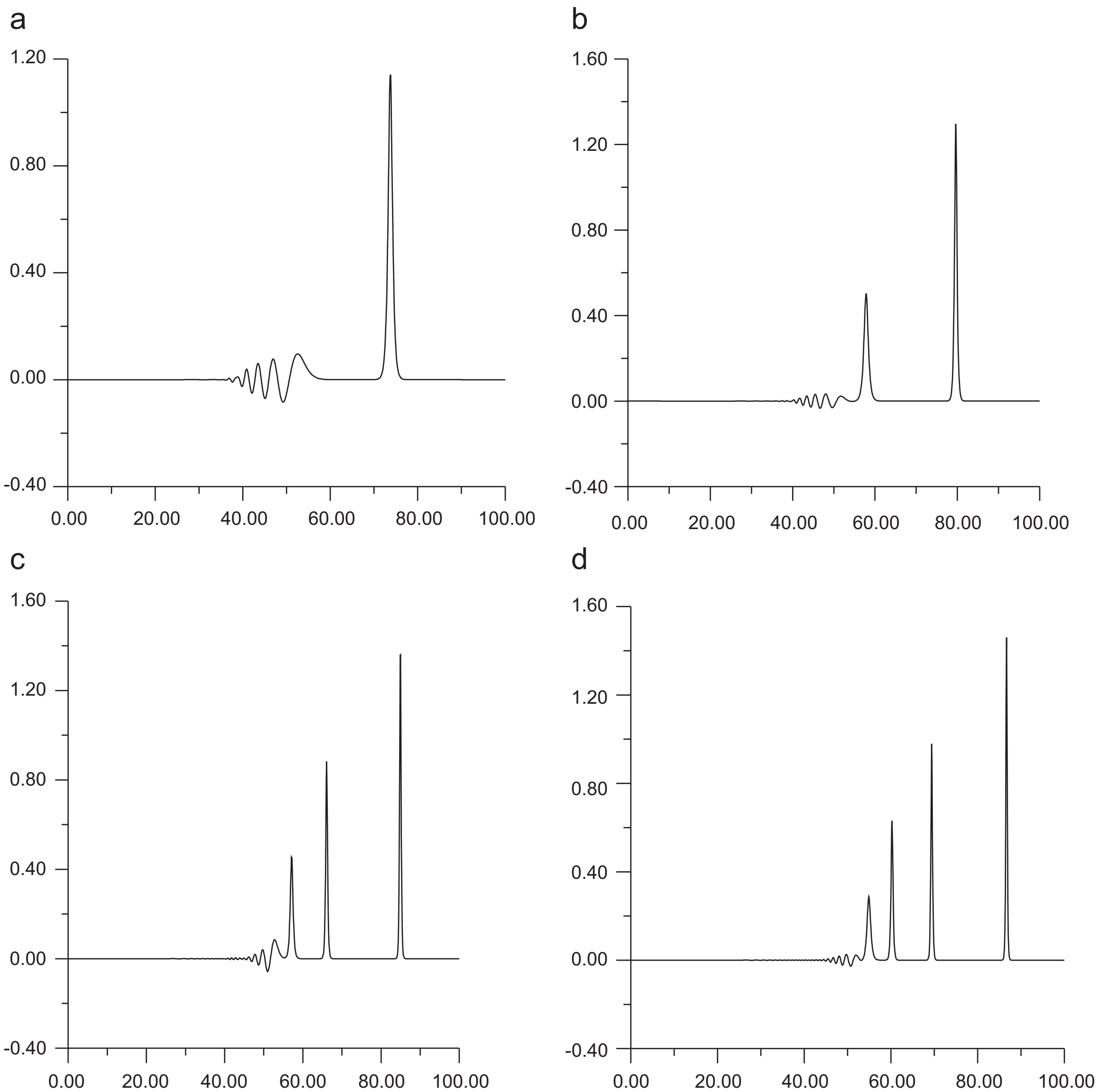

Fig. 5. The Maxwellian initial condition at $t = 14.5$ with (a) $\mu = 0.1$, (b) $\mu = 0.04$, (c) $\mu = 0.015$, (d) $\mu = 0.01$.

for this case are $I_1 = 14.9801$, $I_2 = 15.8218$ and $I_3 = 22.9923$. We find from our numerical scheme, that the invariants I_1, I_2 and I_3 for the interaction of these solitary waves are sensible constants, despite their big amplitudes. The changes are 1.02×10^{-2}, 3×10^{-1} and 2.41×10^{-1} percent, respectively, for the computer run and the results are recorded in Table 6. Fig. 4 shows the details of interaction of these solitary waves at different time levels, and the simulation is done up to $t = 45$.

4.4. The Maxwellian initial condition

In final series of numerical experiments, the development of the Maxwellian initial condition

$$u(x, 0) = \exp(-(x - 40)^2), \tag{28}$$

Table 7
Invariants of MRLW equation using the Maxwellian initial condition

μ	Time	I_1	I_2	I_3
0.1	3	1.77247	1.37767	0.762124
	6	1.77247	1.37762	0.762183
	9	1.77247	1.37765	0.762120
	12	1.77247	1.37763	0.762172
	15	1.77247	1.37764	0.762126
0.04	3	1.77255	1.30072	0.840671
	6	1.77254	1.30073	0.840650
	9	1.77254	1.30073	0.840648
	12	1.77254	1.30074	0.840646
	15	1.77254	1.30074	0.840643
0.015	3	1.77303	1.26703	0.884914
	6	1.77302	1.26702	0.884863
	9	1.77303	1.26702	0.884895
	12	1.77303	1.26704	0.884998
	15	1.77303	1.26706	0.885111
0.01	3	1.77363	1.26058	0.904807
	6	1.77365	1.26079	0.904356
	9	1.77361	1.26039	0.904040
	12	1.77363	1.26063	0.905157
	15	1.77365	1.26076	0.906078

Table 8
Solitary wave generated from a Maxwellian initial condition [7]

μ	Number of solitary waves
0.5	owp[a]
0.1	1
0.05	1
0.04	2
0.03	2
0.02	2
0.015	3
0.01	4
0.005	4

[a] owp: oscillating wave packet.

into a train of solitary waves is examined. We apply it to the problem for different cases: (I) $\mu = 0.1$, (II) $\mu = 0.04$, (III) $\mu = 0.015$ and (IV) $\mu = 0.01$. When μ is reduced, more and more solitary waves are formed. For case (I), only single soliton is generated as shown in Fig. 5a , and for case (II), the Maxwellian pulse breaks up into a train of at least two solitary waves as shown in Fig. 5b. Also for case (III), three stable solitons are generated as shown in Fig. 5c. Finally, for the fourth case, the Maxwellian initial condition has decayed into four stable solitary waves as shown in Fig. 5d. The peaks of the well developed wave lie on a straight line so that their velocities are linearly dependent on their amplitudes and we observe a small oscillating tail appearing behind the last wave in all Maxwellian figures. All states at $t = 14.5$. Moreover, the total number of the solitary waves which are generated from the Maxwellian initial condition according to the results obtained from the numerical scheme in test problem like as in Table 8, can be found in [7]. The values of the quantities I_1, I_2 and I_3 for the cases: $\mu = 0.1, 0.04, 0.015$ and 0.01 are given in Table 7, since for $\mu = 0.1$, the variation of invariants I_2 and I_3 from initial variants changes less than 5×10^{-5} and 6.3×10^{-5} percent, respectively, whereas, the changes in invariant I_1 approach to zero throughout, and for $\mu = 0.04$, the invariants are changed by less than 1×10^{-5}, 2×10^{-5} and 2.8×10^{-5} percent, respectively (see Table 8). For the case $\mu = 0.015$, the invariants I_1,

I_2 and I_3 are changed by less than 1×10^{-5}, 4×10^{-5} and 2.48×10^{-4} percent, respectively. Finally, for $\mu = 0.01$, we find that the changes in these invariants are less than 4×10^{-5} and 4×10^{-4} and 1.722×10^{-3} percent, respectively, during the time of running, and the simulations are done up to $t = 15$.

5. Conclusion

In this paper, the collocation method using cubic B-splines was applied to study the solitary waves of the MRLW equation, and it is shown that our scheme is marginally stable, and more accurate than other cubic collocation schemes in [7]. We tested our scheme through single solitary wave in which the analytic solution is known, and then extended it to study the interaction of solitons where no analytic solution is known during the interaction. The Maxwellian initial condition was also used. Moreover, despite the fact that the wave does not change, results show that the interaction results in a tail of small amplitude in two and clearly three soliton interactions, and the conservation laws were reasonably satisfied. The appearance of such a tail can be beneficial in further study.

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